Lecture 1:

Image transformation

Let $\mathcal{I}$ = Collection of images of size N and range of intensity $[0, M]$.

$$= \{ f \in M_{N \times N}(\mathbb{R}) : 0 \leq f(i,j) \leq M ; 1 \leq i, j \leq N \}$$

(for simplicity, assume f is a square image; can be general $N_1 \times N_2$ image)

Image transformation = $\mathcal{O} : \mathcal{I} \to \mathcal{I}$ (transform one image to another)

Imaging problems

① Find a suitable transformation $T \Rightarrow$ $g := T(f)$ becomes good!

noisy image

② Given a noisy/distorted image g and transformation T, find original clean image f.

unknown

$$g = T(f) + n$$

known known unknown (Inverse problem)

Definition: (Linear image transformation)

$\mathcal{O}: \mathcal{I} \to \mathcal{I}$ is linear $\iff$ $\mathcal{O}(af + g) = a\,\mathcal{O}(f) + \mathcal{O}(g)$ for $\forall f, g \in \mathcal{I}$; $\forall a \in \mathbb{R}$

Take $f \in \mathcal{I}$. Let

$$f = \begin{pmatrix} f(1,1) & \cdots & f(1,N) \\ f(2,1) & & f(2,N) \\ \vdots & f(i,j) & \vdots \\ f(N,1) & \cdots & f(N,N) \end{pmatrix} = \sum_{i=1}^{N} \sum_{j=1}^{N} \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & f(i,j) & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}$$

Let $g = \mathcal{O}(f)$. Assume $\mathcal{O}$ is linear, then:

$$g(\alpha, \beta) = \left[\sum_{x=1}^{N} \sum_{y=1}^{N} \mathcal{O}\left(\begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & f(x,y) & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \right) \right]_{\alpha,\beta}$$

$$= \sum_{x=1}^{N} \sum_{y=1}^{N} f(x,y)\, h(x, \alpha, y, \beta) \quad \text{where}$$

$$h(x, \alpha, y, \beta) = \left[\mathcal{O}(P_{xy}) \right]_{\alpha,\beta} \; ; \; P = \begin{array}{cc} \begin{array}{ccccc} & & y^{th} & & \\ & & \downarrow & & \end{array} & \\ \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} & \leftarrow x^{th} \end{array}$$

Remark: $h(x, \alpha, y, \beta)$ = how much input value at (x, y) influence the output value at (α, β).

Pixel (x,y) affecting pixel (α, β) by a weight $h(x, \alpha, y, \beta)$.

Definition: (Point spread function)

$h(\cdot, \alpha, \cdot\, \beta)$ is called the PSF at (α, β).

Fix α, β. Let x, y as variables!

Definition: (Shift-invariant)

Shift invariant $\Leftrightarrow$ $h(x, \alpha, y, \beta) = \tilde{h}(\alpha - x, \beta - y)$ for $\forall\, 1 \leq x, y, \alpha, \beta \leq N$

Definition: (Convolution) Let $f, g \in \mathcal{I}$.

Convolution of f and g $\Leftrightarrow$ $f * g(\alpha, \beta) \overset{(def)}{=} \sum_{x=1}^{N} \sum_{y=1}^{N} f(x, y)\, g(\alpha - x, \beta - y)$

(Assume f and g are periodically extended: $\begin{cases} f(x+iN, y+jN) = f(x,y) \\ g(x+iN, y+jN) = g(x,y) \end{cases}$ $\forall\, i, j \in \mathbb{Z}$)

Theorem: PSF is shift-invariant $\Rightarrow$ the operator $\mathcal{O}$ is a convolution with the input image.

Proof: Let $g := \mathcal{O}(f)$. $g(\alpha, \beta) = \sum_{x=1}^{N} \sum_{y=1}^{N} f(x,y) \underbrace{h(x, \alpha, y, \beta)}_{\tilde{h}(\alpha - x,\, \beta - y)}$

$= f * \tilde{h}(\alpha, \beta)$

Remark:

- $f * h = h * f$ (exercise)
- Convolution is important for understanding image blur.

Definition: (Separable)

Separable $\Longleftrightarrow h(x, \alpha, y, \beta) = h_c(x, \alpha)\, h_r(y, \beta)$ for $\forall \quad 1 \leq x, y, \alpha, \beta \leq N.$